

Operations on Arc Diagrams and Degenerations for Invariant Subspaces of Linear Operators. Part II

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Dedicated to Professor Daniel Simson on the occasion of his 75th birthday

Abstract

For a partition β , denote by N_β the nilpotent linear operator of Jordan type β . Given partitions β, γ , we investigate the representation space ${}_2\mathbb{V}_\gamma^\beta$ of all short exact sequences

$$\mathcal{E} : 0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$$

where α is any partition with each part at most 2. Due to the condition on α , the isomorphism type of a sequence \mathcal{E} is given by an arc diagram Δ ; denote by \mathbb{V}_Δ the subset of ${}_2\mathbb{V}_\gamma^\beta$ of all sequences isomorphic to \mathcal{E} .

Thus, the space ${}_2\mathbb{V}_\gamma^\beta$ carries a stratification given by the subsets of type \mathbb{V}_Δ . We compute the dimension of each stratum and show that the boundary of a stratum \mathbb{V}_Δ consists exactly of those $\mathbb{V}_{\Delta'}$ where Δ' is obtained from Δ by a non-empty sequence of arc moves of five possible types **(A)** – **(E)**.

The case where all three partitions are fixed has been studied in [3] and [4]. There, arc moves of types **(A)** – **(D)** suffice to describe the boundary of a \mathbb{V}_Δ in $\mathbb{V}_{\alpha,\gamma}^\beta$. Our fifth move **(E)**, “explosion”, is needed to break up an arc into two poles to allow for changes in the partition α .

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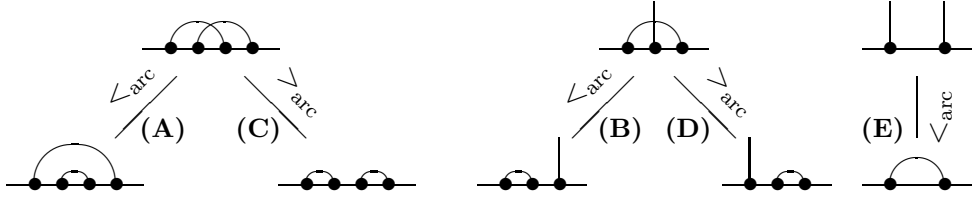
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1. Introduction

In this paper we generalize results presented in [3] and [4] where arc diagrams were applied to investigate the degeneration order for a certain class of invariant subspaces of nilpotent linear operators. We enlarge the set of arc-moves introduced in [3] and compare the partial order induced by these moves with the partial order given by degenerations in a variety associated with invariant subspaces of nilpotent linear operators.

1.1. Arc diagrams and the arc-order

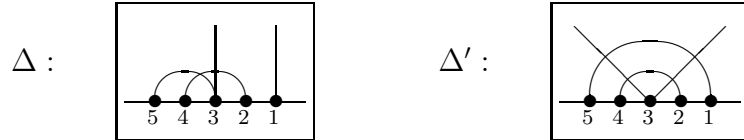
Two diagrams of arcs and poles are said to be in *arc-order* if the first is obtained from the second by a sequence of moves of type



If the arc diagrams Δ and Δ' are in relation, we write $\Delta \leq_{\text{arc}} \Delta'$. Since each move either decreases the number of crossings or the number of poles, \leq_{arc} is a partial order. The arc-moves (A), (B), (C) and (D) were introduced in [3] and the arc-move (E) is new.

Formally, an arc diagram is a finite set of arcs and poles in the Poincaré half plane. We assume that all end points are natural numbers (arranged from right to left) and permit multiple arcs and poles. In addition, each natural number can be marked by one or several circles (which are not affected by any of the arc moves).

Example: The diagrams Δ and Δ' are in arc-order via a single move of type (B).



1.2. Invariant subspaces of nilpotent linear operators

Let k be a field. We call a $k[T]$ -module

$$(1.1) \quad N_\alpha = N_\alpha(k) = \bigoplus_{i=1}^s k[T]/(T^{\alpha_i}),$$

where $\alpha = (\alpha_1 \geq \dots \geq \alpha_s)$ is a partition, a nilpotent linear operator. Thus, N_α is a representation of the quiver

$$\varphi \circlearrowleft$$

subject to the relation $\varphi^{\alpha_1} = 0$. By \mathcal{S} we denote the category of all representations M of the quiver

$$Q : \quad \varphi' \circlearrowleft_1 \xrightarrow{\psi} \circlearrowright_2 \varphi$$

subject to the following conditions: The endomorphisms $M(\varphi')$ and $M(\varphi)$ are nilpotent; the linear map $M(\psi)$ is a monomorphism; and the relation $\varphi\psi = \psi\varphi'$ is satisfied. Equivalently, \mathcal{S} is the category of all monomorphisms between nilpotent linear operators, or, the category of all short exact sequences of nilpotent linear operators.

Given three partitions α, β, γ , we denote by $\mathcal{S}_{\alpha, \gamma}^\beta$ the full subcategory of \mathcal{S} consisting of all short exact sequences $0 \rightarrow N_\alpha \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$ and consider the full subcategories of \mathcal{S} with objects:

$$(1.2) \quad \mathcal{S}_\gamma^\beta = \bigcup_{\alpha} \mathcal{S}_{\alpha, \gamma}^\beta, \quad {}_2\mathcal{S}_\gamma^\beta = \bigcup_{\alpha; \alpha_1 \leq 2} \mathcal{S}_{\alpha, \gamma}^\beta.$$

We assign to each object M in ${}_2\mathcal{S}_\gamma^\beta$ an arc diagram $\Delta(M)$ which depends only on the isomorphism type of M . By ${}_2\mathcal{D}_\gamma^\beta$ we denote the set of all arc diagrams of type (β, γ) . Given $\Delta \in {}_2\mathcal{D}_\gamma^\beta$, we denote by $\alpha(\Delta)$ the partition which contains a part 2 for each arc, and a part 1 for each pole. Thus, a short exact sequence with arc diagram Δ has the form $0 \rightarrow N_{\alpha(\Delta)} \rightarrow N_\beta \rightarrow N_\gamma \rightarrow 0$.

Let k be an algebraically closed field, let $d = (a, b) \in \mathbb{N}^2$ be a dimension vector with $b \geq a$, and let β and γ be partitions of length b and $b - a$, respectively.

In this paper we are interested in the subspace ${}_2\mathbb{V}_\gamma^\beta$ of the representation space $\text{rep}_d(Q)$ consisting of all representations which are isomorphic to an object in ${}_2\mathcal{S}_\gamma^\beta$.

The space ${}_2\mathbb{V}_\gamma^\beta$ is invariant under the operation of the algebraic group $G = Gl_a(k) \times Gl_b(k)$ acting on $\text{rep}_d(Q)$; the G -orbits in ${}_2\mathbb{V}_\gamma^\beta$ are in one-to-one correspondence with the isomorphism classes in ${}_2\mathcal{S}_\gamma^\beta$. For $\Delta \in {}_2\mathcal{D}_\gamma^\beta$ we denote by \mathbb{V}_Δ the G -orbit of all representations which have arc diagram Δ .

1.3. Main results

The arc-diagrams of invariant subspaces give a stratification of ${}_2\mathbb{V}_\gamma^\beta$:

$${}_2\mathbb{V}_\gamma^\beta = \bigcup_{\Delta} \mathbb{V}_\Delta.$$

More precisely, we have:

PROPOSITION 1.3. *The set $\{\mathbb{V}_\Delta : \Delta \in \mathcal{D}\}$ where $\mathcal{D} = {}_2\mathcal{D}_\gamma^\beta$ forms a stratification for $\mathbb{V} = {}_2\mathbb{V}_\gamma^\beta$ in the sense that*

1. *each \mathbb{V}_Δ is locally closed in \mathbb{V} ;*
2. *\mathbb{V} is the disjoint union $\bigcup_{\Delta \in \mathcal{D}}^\bullet \mathbb{V}_\Delta$;*
3. *and for each Δ there is a finite subset $U_\Delta \subset \mathcal{D}$ such that the closure $\overline{\mathbb{V}_\Delta}$ is just the union $\bigcup_{\Gamma \in U_\Delta} \mathbb{V}_\Gamma$.*

Moreover an arc diagram determines the dimension of its stratum as follows.

THEOREM 1.4. *Let k be an algebraically closed field, and let β, γ be partitions. Suppose the arc diagram $\Delta \in {}_2\mathcal{D}_\gamma^\beta$ has $x(\Delta)$ crossings. Then*

$$\dim \mathbb{V}_\Delta = |\beta|^2 + |\alpha|^2 - n(\alpha) - n(\beta) - n(\gamma) - |\beta| - x(\Delta),$$

where $\alpha = \alpha(\Delta)$ and where $n(\lambda) = \sum_i \lambda_i(i-1)$ denotes the **moment** and $|\lambda| = \lambda_1 + \lambda_2 + \dots$ - the length of a partition λ .

The second main result of this paper describes the boundary of each stratum.

THEOREM 1.5. *Suppose that k is an algebraically closed field and that β, γ are partitions. For arc diagrams $\Gamma, \Delta \in {}_2\mathcal{D}_\gamma^\beta$ we have*

$$\Gamma \in U_\Delta \quad \text{if and only if} \quad \Delta \leq_{\text{arc}} \Gamma.$$

In the proof of the “only if” part, we use a version of an algorithm from [3] which delivers a sequence of arc moves that convert Γ to Δ .

Remark: 1. In [3] we investigated arc- and deg-orders in sets $\mathbb{V}_{\alpha,\gamma}^\beta(k)$, where $\alpha_1 \leq 2$. In this paper we work in a bigger set ${}_2\mathbb{V}_\gamma^\beta$ and prove similar results without fixing the partition α . Moreover, in Section 8 we present examples showing that we have to fix the partitions β and γ .

2. Working with arc-diagrams, we have to assume that $\alpha_1 \leq 2$, because otherwise we do not have the natural bijection between arc diagrams and orbits of our group action.
3. In the paper we investigate the degeneration order induced by this group action and compare this order with the arc-order and with two algebraic orders (hom-order and ext-order), see Section 3.1 for definitions.

1.4. Organization of this paper

The paper is organized as follows.

- In Section 2 we introduce terminology and describe a bijection between arc diagrams and isomorphism types of objects in the category ${}_2\mathcal{S}_\gamma^\beta$.
- In Section 3 we recall the definition of the partial orders \leq_{arc} , \leq_{ext} , \leq_{deg} and \leq_{hom} . Moreover, in Theorem 3.1 we discuss relations between them.
- In Section 4 we give a brief exposition of the category $\mathcal{S}_2(k)$.
- Some technical facts for later use are presented in Section 5.
- Section 6 contains the proof of the implication

$$\leq_{\text{hom}} \implies \leq_{\text{arc}}$$

that is the crucial part of the proof of Theorem 1.5. Our proof uses methods derived from [3].

- In Section 7 some combinatorial properties of the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ are presented.

- In Theorems 1.5 and 3.1 we assume that two partitions β and γ are fixed. Some relevant counterexamples indicated in Section 8 show that this assumption is necessary.
- Section 9 contains a proof of Theorem 1.4.

2. Arc diagrams and invariant subspaces

For fixed partitions $\gamma \subseteq \beta$, there is a bijection between the set of arc diagrams and the set of isomorphism classes of objects in the category ${}_2\mathcal{S}_\gamma^\beta$ (1.2). In this section we describe this bijection.

In [3] Littlewood-Richardson tableaux and Klein tableaux were applied to associate an arc diagram with an object in \mathcal{S}_2 , where $\mathcal{S}_2 = \bigcup_{\gamma \subseteq \beta} {}_2\mathcal{S}_\gamma^\beta$. For the definition of a Klein tableau for an object in \mathcal{S}_2 we refer to [2] or [7].

2.1. Invariant subspaces

Let k be an arbitrary field. By \mathcal{N} or $\mathcal{N}(k)$ we denote the category of all nilpotent linear operators N_α (1.1). We write the objects of \mathcal{N} as pairs (V, φ) where V is the underlying k -vector space and $\varphi : V \rightarrow V$ the nilpotent k -linear endomorphism given by multiplication by T . If (V, φ) , (V', φ') are objects in \mathcal{N} , then a morphism $f : (V, \varphi) \rightarrow (V', \varphi')$ in \mathcal{N} is a linear map $f : V \rightarrow V'$ such that $\varphi'f = f\varphi$. Let

$$(2.1) \quad N_\alpha = (k^{|\alpha|}, \varphi_\alpha),$$

where φ_α is given by the nilpotent block matrix consisting of Jordan blocks of type α .

Denote by \mathcal{S} the category of all systems $f = (N_\alpha, N_\beta, f)$, where f is a monomorphism. For $f = (N_\alpha, N_\beta, f)$ and $g = (N_{\alpha'}, N_{\beta'}, g)$ a morphism $H : f \rightarrow g$ is a pair (h_1, h_2) of homomorphisms $h_1 : N_\alpha \rightarrow N_{\alpha'}$ and $h_2 : N_\beta \rightarrow N_{\beta'}$ such that

$$g \circ h_1 = h_2 \circ f.$$

For a natural number n , we write \mathcal{S}_n or $\mathcal{S}_n(k)$ for the full subcategory of \mathcal{S} of all systems where the operator acts on the subspace with nilpotency index at most n . Thus, the objects in \mathcal{S}_2 are the systems (N_α, N_β, f) where $\alpha_1 \leq 2$.

2.2. Pickets and bipickets

The category $\mathcal{S}_2(k)$ is of particular interest for us in this paper. Each indecomposable object is either isomorphic to a *picket* that is, it has the form



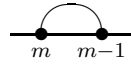
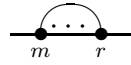
$$P_\ell^m = (N_{(\ell)}, N_{(m)}, \iota)$$

where $0 \leq \ell \leq \min\{2, m\}$ (so the ambient space $N_{(m)}$ has only one Jordan block, and $N_{(\ell)}$ is the unique T -invariant subspace of dimension ℓ), or to a *bipicket*

$$B_2^{m,r} = (N_{(2)}, N_{(m,r)}, \delta)$$

where $1 \leq r \leq m-2$ and $\delta : k[T]/(T^2) \rightarrow k[T]/(T^m) \oplus k[T]/(T^r)$ is given by $\delta(1) = (T^{m-2}, T^{r-1})$. Whenever we want to emphasize the dependence on the field k , we will write $P_\ell^m = P_\ell^m(k)$ and $B_2^{m,r} = B_2^{m,r}(k)$.

Thanks to this classification we can associate with any object in $\mathcal{S}_2(k)$ an arc diagram and partition (β, γ) type as follows. First we list some arc diagrams in the following table.

| Arc diagrams for the objects in $\text{ind}\mathcal{S}_2$ | | | | | |
|---|----------------|---|---|---|---|
| $X :$ | P_0^m | P_1^m | P_2^m | $P_2^m \oplus P_0^{m-1}$ | $B_2^{m,r}$ |
| $\Delta(X) :$ | \emptyset |  |  |  |  |
| partition (β, γ) | $\beta = (m)$ | $\beta = (m)$ | $\beta = (m)$ | $\beta = (m, m-1)$ | $\beta = (m, r)$ |
| type of X | $\gamma = (m)$ | $\gamma = (m-1)$ | $\gamma = (m-2)$ | $\gamma = (m-1, m-2)$ | $\gamma = (m-1, r-1)$ |

The arc diagram for an object X in \mathcal{S}_2 is created in the following way. We decompose

$$X = Y \oplus (P_2^{s_1} \oplus P_0^{s_1-1}) \oplus \dots \oplus (P_2^{s_r} \oplus P_0^{s_r-1}) \oplus P_2^{m_1} \oplus \dots \oplus P_2^{m_k} \oplus P_0^{w_1} \oplus \dots \oplus P_0^{w_l},$$

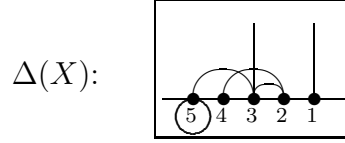
where Y does have neither summands of type P_2^m nor of type P_0^m , and $w_1, \dots, w_l \notin \{m_1-1, \dots, m_k-1\}$. For any indecomposable summand of Y and any summand $(P_2^{s_1} \oplus P_0^{s_1-1}), \dots, (P_2^{s_r} \oplus P_0^{s_r-1}), P_2^{m_1}, \dots, P_2^{m_k}, P_0^{w_1}, \dots, P_0^{w_l}$ we create the arc diagram using the above table. The arc diagram $\Delta(X)$ of X is the union of all these arc diagrams.

Similarly we define the *type* of X as the union of partitions β and γ described in the table above.

Example: For the object

$$X = B_2^{5,3} \oplus B_2^{4,2} \oplus P_2^5 \oplus P_0^2 \oplus P_2^3 \oplus P_1^3 \oplus P_0^1 \oplus P_1^1,$$

the arc diagram is



and the type of X is $(\beta, \gamma) = ((5, 5, 4, 3, 3, 3, 2, 2, 1, 1), (4, 3, 3, 2, 2, 2, 1, 1, 1))$.

Remark: Note that two non-isomorphic object in \mathcal{S}_2 can have the same arc diagram, e.g. $B_2^{5,2}$ and $B_2^{5,2} \oplus P_0^3$. However, if we fix partitions γ and β , the arc diagram uniquely determines the object it is associated with.

Definition: We denote by ${}_2\mathcal{D}_\gamma^\beta$ the set of all arc diagrams which arise from objects in ${}_2\mathcal{S}_\gamma^\beta$.

We obtain as a consequence:

THEOREM 2.2. *For any field k and for fixed partitions $\gamma \subseteq \beta$, there is a one-to-one correspondence between the set of isomorphism classes of objects in ${}_2\mathcal{S}_\gamma^\beta$ and the set of arc diagrams in ${}_2\mathcal{D}_\gamma^\beta$.* \square

Definition: We say two invariant subspaces $Y, Z \in {}_2\mathcal{S}_\gamma^\beta$ are in *arc-order*, in symbols $Y \leq_{\text{arc}} Z$, if $\Delta(Y) \leq_{\text{arc}} \Delta(Z)$ holds.

3. The degeneration order

Assume that k is an algebraically closed field. For natural numbers $a \leq b$ the representation space of the quiver Q corresponding to the dimension vector (a, b) is the affine variety

$${}^a\mathbb{V}^b = \mathbb{M}_a(k) \times \mathbb{M}_{a \times b}(k) \times \mathbb{M}_b(k),$$

where $\mathbb{M}_{a \times b}(k)$ is the set of $a \times b$ matrices with coefficients in k and $\mathbb{M}_a(k) = \mathbb{M}_{a \times a}(k)$. We work with the Zariski topology and with the induced topology for all subsets of ${}^a\mathbb{V}^b$. Fix partitions $\gamma \subseteq \beta$ such that $|\beta| = b$ and $|\gamma| = b - a$. We define \mathbb{V}_γ^β as the subset of ${}^a\mathbb{V}^b$ consisting of all points $F = (f_1, f, f_2)$,

such that $f_1^a = 0$, $f_2^b = 0$, $ff_1 - f_2f = 0$, f_2 has the Jordan type β , f has maximal rank and $\text{Coker } f$ has type γ . Moreover, by ${}_2\mathbb{V}_\gamma^\beta$ denote the subset of \mathbb{V}_γ^β consisting of all points (f_1, f, f_2) such that f_1 has nilpotency index less than or equal to 2, that is, the Jordan blocks in the Jordan normal form for f_1 have size at most 2. On \mathbb{V}_γ^β (resp. ${}_2\mathbb{V}_\gamma^\beta$), there acts the algebraic group $G = \text{Gl}(a, b) = \text{Gl}(a) \times \text{Gl}(b)$ via $(g, h) \cdot (f_1, f, f_2) = (gf_1g^{-1}, hfg^{-1}, hf_2h^{-1})$. For a point $F \in \mathbb{V}_\gamma^\beta$, denote by \mathcal{O}_F the orbit of F under action of G .

Remark: Note that the G -orbits in \mathbb{V}_γ^β (resp. in ${}_2\mathbb{V}_\gamma^\beta$) are in 1-1-correspondence with the isomorphism classes of objects in \mathcal{S}_γ^β (resp. ${}_2\mathcal{S}_\gamma^\beta$).

Definition: Suppose that the points $F, H \in {}_2\mathbb{V}_\gamma^\beta(k)$ correspond to objects $Y = (N_\alpha, N_\beta, f)$ and $Z = (N_{\tilde{\alpha}}, N_\beta, h)$ in ${}_2\mathcal{S}_\gamma^\beta(k)$.

The relation $Y \leq_{\text{deg}} Z$ is defined to hold if and only if $\mathcal{O}_H \subseteq \overline{\mathcal{O}_F}$ where $\overline{\mathcal{O}_F}$ is the closure of \mathcal{O}_F in ${}_2\mathbb{V}_\gamma^\beta(k)$.

3.1. The algebraic orders

Our aim is to prove that the degeneration order and the arc order are equivalent. In the proof we use the following classical algebraic orders. Let $Y = (N_\alpha, N_\beta, f)$ and $Z = (N_{\tilde{\alpha}}, N_\beta, g)$ be objects in $\mathcal{S}_\gamma^\beta(k)$.

- The relation $Y \leq_{\text{ext}} Z$ holds if there exist a natural number s , objects M_i, U_i, V_i in $\mathcal{S}(k)$ and short exact sequences $0 \rightarrow U_i \rightarrow M_i \rightarrow V_i \rightarrow 0$ in $\mathcal{S}(k)$ such that $Y \cong M_1$, $U_i \oplus V_i \cong M_{i+1}$ for $1 \leq i \leq s$, and $Z \cong M_{s+1}$.
- The relation $Y \leq_{\text{hom}} Z$ holds if

$$[X, Y] \leq [X, Z]$$

for any object X in $\mathcal{S}(k)$. Here we write $[X, Y] = \dim_k \text{Hom}_{\mathcal{S}}(X, Y)$ for \mathcal{S} -modules X, Y . By [3, Lemma 3.3], the relation $Y \leq_{\text{hom}} Z$ holds if $[X, Y] \leq [X, Z]$ for every object X in $\mathcal{S}_2(k)$.

3.2. The partial orders are equivalent

We can present the proof of Theorem 1.5, up to two results about the arc-order which are shown in the next section.

We restate the theorem to include statements about arbitrary fields.

THEOREM 3.1. *Let k be an arbitrary field and assume that $Y, Z \in {}_2\mathcal{S}_\gamma^\beta$. The following conditions are equivalent*

1. $Y \leq_{\text{arc}} Z$,
2. $Y \leq_{\text{ext}} Z$,
3. $Y \leq_{\text{hom}} Z$.

If in addition the field k is algebraically closed, then the conditions stated above are equivalent with

- (4) $Y \leq_{\text{deg}} Z$.

Proof. Applying the functor $\text{Hom}_k(X, -)$ to the short exact sequences given in the definition of \leq_{ext} , it is easy to see that $Y \leq_{\text{ext}} Z$ implies $Y \leq_{\text{hom}} Z$. If k is an algebraically closed field, then by [1, 5] (see [3, Section 3] for details), we have

$$Y \leq_{\text{ext}} Z \implies Y \leq_{\text{deg}} Z \implies Y \leq_{\text{hom}} Z.$$

For any field k , the implications

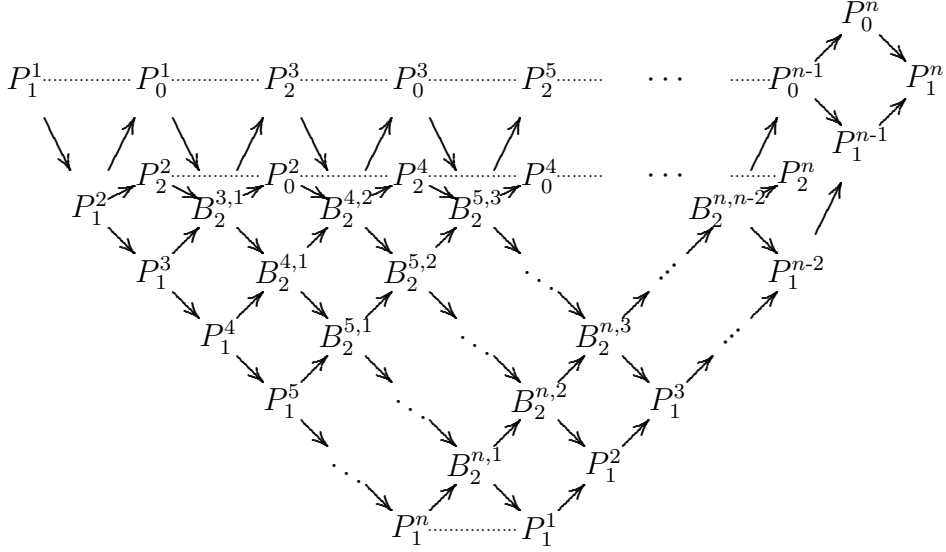
$$Y \leq_{\text{hom}} Z \implies Y \leq_{\text{arc}} Z \implies Y \leq_{\text{ext}} Z$$

follow from Theorem 6.1 and Lemma 4.1, respectively. □

4. The category $\mathcal{S}_2(k)$

In this section we shortly recall properties of the category $\mathcal{S}_2(k)$. Denote by $\mathcal{S}_2^n(k)$ the full subcategory of $\mathcal{S}_2(k)$ of all objects where the operator acts with nilpotency index at most n on the ambient space. We have seen in [6, Section 3.2] that $\mathcal{S}_2^n(k)$ is an exact Krull-Remak-Schmidt category with Auslander-Reiten sequences.

The Auslander-Reiten quiver for each of the categories $\mathcal{S}_2^n(k)$ is obtained by identifying the objects of type P_1^r on the left with their counterparts on the right in the following picture, thus yielding a Moebius band.



For each pair (X, Y) of indecomposable objects in $\mathcal{S}_2(k)$ we determine in the table below the dimension of the k -space $\text{Hom}_{\mathcal{S}}(X, Y)$, see [7, Lemma 4] and [3].

| Dimensions of Spaces $\text{Hom}(X, Y)$, $X, Y \in \text{ind}\mathcal{S}_2(k)$ | | | | |
|---|--|----------------------------------|--|--------------------------------------|
| X | $Y = P_0^m$ | P_2^m | $B_2^{m,r}$ | P_1^m |
| P_0^ℓ | $\min\{\ell, m\}$ | $\min\{\ell, m\}$ | $\min\{\ell, m\} + \min\{\ell, r\}$ | $\min\{\ell, m\}$ |
| P_2^ℓ | $\min\{\ell - 2, m\}$ | $\min\{\ell, m\}$ | $\min\{\ell - 1, m\} + \min\{\ell - 1, r\}$ | $\min\{\ell - 1, m\}$ |
| $B_2^{\ell,t}$ | $\min\{\ell - 1, m\} + \min\{t - 1, m\}$ | $\min\{\ell, m\} + \min\{t, m\}$ | $\min\{\ell - 1, m\} + \min\{t, m\} + \min\{\ell - 1, r\} + \min\{t, r\} - \mathbf{1}\{\ell > m \text{ and } t \leq r\}$ | $\min\{\ell - 1, m\} + \min\{t, m\}$ |
| P_1^ℓ | $\min\{\ell - 1, m\}$ | $\min\{\ell, m\}$ | $\min\{\ell, m\} + \min\{\ell - 1, r\}$ | $\min\{\ell, m\}$ |

We denote by $\mathbf{1}$ the *characteristic function* corresponding to the property specified in parantheses.

For the sake of simplicity we use the notation

$$B_2^{m,m-1} = P_2^m \oplus P_0^{m-1}$$

for $m \geq 2$. We observe that the notation is consistent with the formulae above.

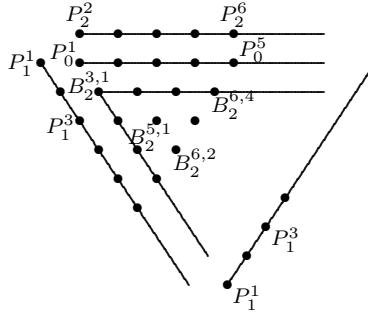
4.1. How operations change the hom spaces

Throughout this section, $Y, Z \in \mathcal{S}_2$ will be objects of the same partition type (β, γ) . Following [3], we introduce two matrices, the multiplicity matrix $\delta M = \delta M(Y, Z)$ and the hom matrix $\delta H = \delta H(Y, Z)$; in each case the indexing set is the set of isomorphism types of indecomposable objects in \mathcal{S}_2 . The matrices are defined as follows:

$$\delta M_X = \mu_X(Z) - \mu_X(Y), \quad \text{and} \quad \delta H_X = [X, Z]_{\mathcal{S}} - [X, Y]_{\mathcal{S}},$$

where $[X, Z]_{\mathcal{S}} = \dim \text{Hom}_{\mathcal{S}}(X, Z)$ and where $\mu_X(Z)$ denotes the number of direct summands of Z that are isomorphic to X .

We visualize the matrices by indicating the value at $X \in \text{ind} \mathcal{S}_2$ in the position of X in the Auslander-Reiten quiver for \mathcal{S}_2^n , with n large enough. We sketch this quiver as follows: The modules on the top line are the P_2^m , those on the second line are the P_0^r ; the modules in the triangle have type $B_2^{m,r}$. The modules P_1^r are repeated twice, on the diagonal at the left and on the antidiagonal at the right.



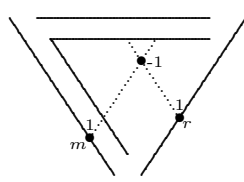
We determine how the matrices change under the operation **(E)** on the arc diagram.

Suppose Z is obtained from Y by a transformation

$$\textbf{(E)} : \quad \begin{array}{c} \text{arc} \\ \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ m \quad r \end{array} \leq_{\text{arc}} \begin{array}{c} \text{---} \bullet \text{---} \text{---} \bullet \text{---} \\ m \quad r \end{array}$$

where $m > r + 1$. Recall that the arc in the diagram for Y represents the bipicket $B_2^{m,r}$ which is replaced by the summands P_1^m and P_1^r in Z that give rise to the corresponding poles in the diagram for Z .

Thus, the multiplicity matrix is as follows.



Note that the marked points correspond to a short exact sequence

$$0 \longrightarrow P_1^m \longrightarrow B_2^{m,r} \longrightarrow P_1^r \longrightarrow 0$$

which proves the implication

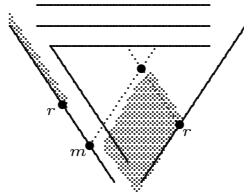
$$Y \leq_{\text{arc}} Z \implies Y \leq_{\text{ext}} Z.$$

Next we determine the Hom matrix $\delta H = \delta H(B_2^{m,r}, P_1^m \oplus P_1^r)$. By Lemma 5.1, we have $\delta H_{P_0^\ell} = 0 = \delta H_{P_2^\ell}$. We compute the remaining numbers using the table given in the first part of Section 4:

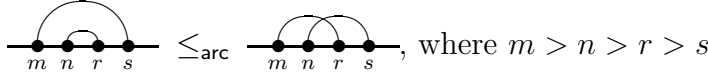
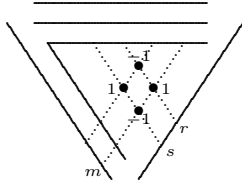
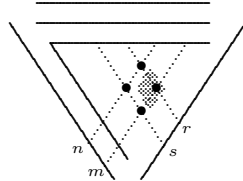
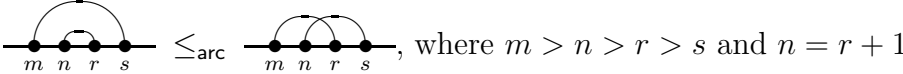
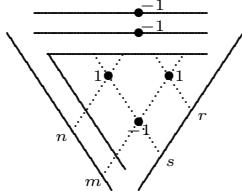
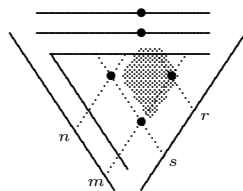
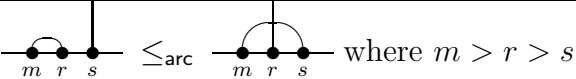
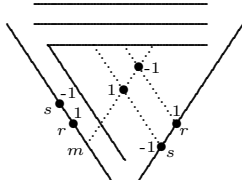
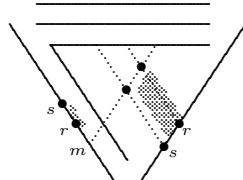
$$\delta H_{P_1^\ell} = 1, \text{ where } \ell \leq r$$

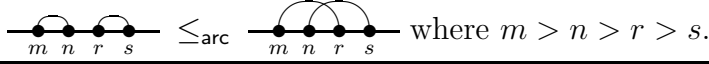
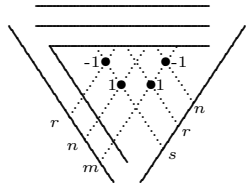
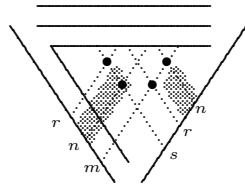
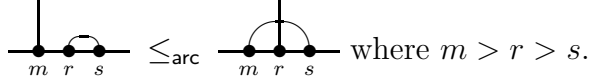
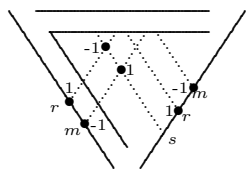
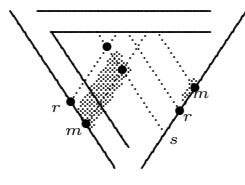
$$\delta H_{B_2^{\ell,t}} = \mathbf{1}\{m < \ell \text{ and } t \leq r\}.$$

Thus, the only indecomposables $X \in \mathcal{S}_2$ for which $\delta H_X \neq 0$ are the $B_2^{\ell,t}$ where $m < \ell$ and $t \leq r$ and P_1^ℓ where $\ell \leq r$. For each such module X we have $\delta H_X = 1$. They lie in the shaded region in the diagram below.



We have seen in [3] how the matrices are changing for the operations **(A)**-**(D)** on the arc diagram. We present these matrices in the following table:

| Matrices for moves (A)-(D) | | | |
|----------------------------|--|---|--|
| (A) |  \leq_{arc} where $m > n > r > s$ | | |
| | $\delta M(Y, Z)$ | $\delta H(Y, Z)$ | |
| |  |  | |
| | $0 \longrightarrow B_2^{n,s} \longrightarrow B_2^{m,s} \oplus B_2^{n,r} \longrightarrow B_2^{m,r} \longrightarrow 0$ | | |
| (A') |  \leq_{arc} where $m > n > r > s$ and $n = r + 1$ | | |
| | $\delta M(Y, Z)$ | $\delta H(Y, Z)$ | |
| |  |  | |
| | $0 \longrightarrow B_2^{n,s} \longrightarrow B_2^{m,s} \oplus P_2^n \oplus P_0^r \longrightarrow B_2^{m,r} \longrightarrow 0$ | | |
| (B) |  \leq_{arc} where $m > r > s$. | | |
| | $\delta M(Y, Z)$ | $\delta H(Y, Z)$ | |
| |  |  | |
| | $0 \longrightarrow B_2^{m,s} \longrightarrow B_2^{m,r} \oplus P_1^s \longrightarrow P_1^r \longrightarrow 0$ | | |

| Matrices for moves (A)-(D) | | |
|----------------------------|--|--|
| (C) |  where $m > n > r > s$. | |
| | $\delta M(Y, Z)$ | $\delta H(Y, Z)$ |
| |  |  |
| | $0 \longrightarrow B_2^{m,r} \longrightarrow B_2^{m,n} \oplus B_2^{r,s} \longrightarrow B_2^{n,s} \longrightarrow 0$ | |
| (D) |  where $m > r > s$. | |
| | $\delta M(Y, Z)$ | $\delta H(Y, Z)$ |
| |  |  |
| | $0 \longrightarrow P_1^r \longrightarrow P_1^m \oplus B_2^{r,s} \longrightarrow B_2^{m,s} \longrightarrow 0$ | |

This completes the proof of the following fact.

LEMMA 4.1. *Let k be an arbitrary field and let $Y, Z \in \mathcal{S}_2(k)$ have the same partition type (β, γ) . If $Y \leq_{\text{arc}} Z$, then $Y \leq_{\text{ext}} Z$. \square*

Note that the Hom matrix determines the multiplicity matrix uniquely. Namely, let A be a non-injective indecomposable object with Auslander-Reiten sequence $0 \rightarrow A \rightarrow \bigoplus B_i \rightarrow C \rightarrow 0$ in \mathcal{S}_2^n . Then the multiplicity of A as a direct summand of Y is given by the contravariant defect $\mu_A(Y) = [A, Y] + [C, Y] - \sum_i [B_i, Y]$.

5. Some technical facts

We recall the following result from [3, Lemma 4.2].

LEMMA 5.1. *Suppose $Y, Z \in \mathcal{S}_2$ have the same partition type (β, γ) .*

1. *The Hom matrix $\delta H(Y, Z)$ has zero entry at each position corresponding to a module P_0^m, P_2^m where $m \in \mathbb{N}$.*
2. *Along each diagonal in the Hom matrix, the entries eventually become constant:*

$$\lim_{m \rightarrow \infty} \delta H(Y, Z)_{B_2^{m,r}} = \delta H(Y, Z)_{P_1^m}$$

The following consequence, which is technical but easy to show, will be used in the proof of Proposition 6.2.

LEMMA 5.2. *Suppose that both $Y, Z \in \mathcal{S}_2^n$ have partition type (β, γ) .*

1. *The nonzero part of the Hom matrix $\delta H(Y, Z)$ is contained in the union of the τ -orbits for $X = B_2^{3,1}, \dots, B_2^{n,1}$; they form a Moebius band, i.e. a quiver of type $\mathbb{Z}\mathbb{A}_{n-2}$ with suitable identifications.*
2. *For each non-injective $A \in \text{ind } \mathcal{S}_2^n$, the entry $\delta M_A = \delta M(Y, Z)_A$ in the multiplicity matrix can be read off from the restriction of the Hom matrix to the Moebius band by a formula of type*

$$\delta M_A = \beta_A + \beta_C - \sum_i \beta_{B_i}$$

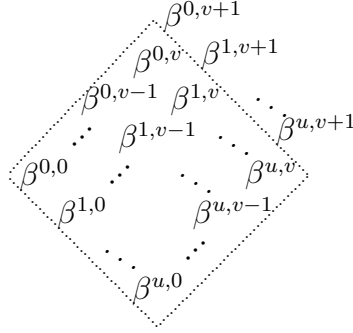
where $0 \rightarrow A \rightarrow \bigoplus_i B_i \rightarrow C \rightarrow 0$ is the Auslander-Reiten sequence starting at A and

$$\beta_X = \begin{cases} \delta H(Y, Z)_X & \text{if } X \text{ is in the Moebius band} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first statement follows from Lemma 5.1, the second from the contravariant defect formula above. \square

In the proof we will use [3, Lemmata 4.6 and 4.7].

LEMMA 5.3. Consider the following matrix of integers.



Suppose the following conditions are satisfied.

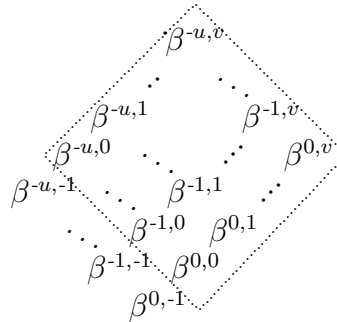
1. All entries are nonnegative
2. The numbers $\beta^{0,0}, \beta^{0,1}, \dots, \beta^{0,v}$ are strictly positive
3. $\beta^{0,v+1} = 0$
4. For each $0 \leq i \leq u-1, 0 \leq j \leq v$,

$$\delta^{ij} = \beta^{i,j} + \beta^{i+1,j+1} - \beta^{i,j+1} - \beta^{i+1,j} \leq 0$$

Then all entries in the parallelogram are positive: $\beta^{i,j} > 0$ for each $0 \leq i \leq u, 0 \leq j \leq v$.

Also the dual version holds:

LEMMA 5.4. Consider the following matrix of integers.



Suppose that in addition to conditions 1 and 2 from the previous lemma also the following are satisfied.

$$3'. \beta^{0,-1} = 0$$

4'. For each $-u \leq i < 0$, $-1 \leq j < v$,

$$\delta^{ij} = \beta^{i,j} + \beta^{i+1,j+1} - \beta^{i,j+1} - \beta^{i+1,j} \leq 0$$

Then all entries in the parallelogram are positive: $\beta^{i,j} > 0$ for each $-u \leq i \leq 0$, $0 \leq j \leq v$. \square

6. The hom-order implies the arc-order

Our aim is to show that $Y \leq_{\text{hom}} Z$ implies $Y \leq_{\text{arc}} Z$.

THEOREM 6.1. *Suppose the objects Y and Z in $\mathcal{S}_2(k)$ have the same partition type (β, γ) . If $Y \leq_{\text{hom}} Z$ holds, then so does $Y \leq_{\text{arc}} Z$.*

This theorem follows from the following.

PROPOSITION 6.2. *Suppose Y and Z have the same partition type (β, γ) , $Y \leq_{\text{hom}} Z$ and $Y \not\cong Z$. Then there is an operation on the arc diagram for Z of type **(A)**, **(B)**, **(C)**, **(D)** or **(E)** which yields a module Z' such that*

$$Z' \leq_{\text{arc}} Z, \quad Z' \not\cong Z, \quad \text{and} \quad Y \leq_{\text{hom}} Z'.$$

Proof. The methods from the proof of [3, Proposition 4.5] can be adapted. Here, we have to consider one more case, namely the move of the type **(E)**. For the convenience of the reader we present the proof with all details. Moreover, after the proof we will illustrate it by an example.

6.1. The proof

The set-up. We assume that the entries in the Hom-matrix $\delta H(Y, Z)$ are all nonnegative and that at least one entry is positive.

The goal. We show that there is a parallelogram in the Hom-matrix (in the shape of one of the shaded regions in Section 4.1) which satisfies the following two conditions.

(P1) All entries within the parallelogram are strictly positive.

(P2) The two indecomposable modules X' and X'' corresponding to the right corner of the parallelogram and to the point just left of the left corner, respectively, occur with higher multiplicity as direct summands of Z than as a direct summand of Y .

Step 1. We choose a number $n > \beta_1$ and work in the category \mathcal{S}_2^n . Recall that in the Hom matrix, the nonzero entries occur in the orbits given by the modules $B_2^{3,1}, \dots, B_2^{n,1}$. The region given by those orbits forms a stripe of type $\mathbb{Z}\mathbb{A}_{n-2}$, with identifications (see Lemma 5.1). For the purpose of this algorithm, we view the Hom matrix as a stripe of type $\mathbb{Z}\mathbb{A}_n$, with the orbits of the $B_2^{3,1}, \dots, B_2^{n,1}$ in the center and two orbits of zeros on the boundary. Pick a sequence $\beta^{0,0}, \dots, \beta^{0,v}$ of positive entries arranged along an anti-diagonal in the Hom matrix, such that the neighboring entries on the anti-diagonal, $\beta^{0,v-1}$ and $\beta^{0,v}$, are both zero. This is possible since the entries in the matrix are nonnegative, since there is at least one positive entry, and since there are zeros on the boundary orbits (see Lemma 5.1).

Step 2. We determine the right corner of the parallelogram. Put $u = 0$. Consider the sequence $\delta^{u,0}, \dots, \delta^{u,v}$ of entries in the multiplicity matrix in the positions given by the $\beta^{u,0}, \dots, \beta^{u,v}$ in the Hom matrix. If one of the entries in the sequence $\delta^{u,0}, \dots, \delta^{u,v}$ is positive, put $u'' = u$ and let $\delta^{u'',w''}$ be the first such entry. In this case, (u'', w'') will be the right corner of the rectangle and X'' , the object corresponding to the position (u'', w'') , will be a summand occurring with higher multiplicity in Z than in Y . We are done with the second step.

If none of the entries in $\delta^{u,0}, \dots, \delta^{u,v}$ is positive, then by Lemma 5.2 we obtain that the assumptions of Lemma 5.3 are satisfied, so using this lemma we obtain that all the numbers $\beta^{u+1,0}, \dots, \beta^{u+1,v}$ in the Hom matrix, on the anti-diagonal just under the previous anti-diagonal, are positive. Put $u := u + 1$ and proceed with the paragraph under Step 2.

Note that this process terminates: For u large enough, $\beta^{u,0}$ will correspond to a point on the boundary of the Hom matrix. Hence $\beta^{u,0} = 0$.

Step 3. We determine the left corner of the parallelogram. Put $u = 0$ and $v = v''$.

Consider the sequence $\delta^{u-1,-1}, \dots, \delta^{u-1,v-1}$ of entries in the multiplicity matrix in the positions just left of the $\beta^{u,0}, \dots, \beta^{u,v}$ in the Hom matrix.

If one of the entries in the sequence $\delta^{u-1,-1}, \dots, \delta^{u-1,v-1}$ is positive, put $u' = u - 1$ and let $\delta^{u',w'}$ be the last such entry. In this case, (u', w') will be the point just left of the left corner of the rectangle and X' , the object

corresponding to the position (u', w') , will be a summand occurring with higher multiplicity in Z than in Y . We are done with the third step.

If none of the entries in $\delta^{u-1, -1}, \dots, \delta^{u-1, v-1}$ is positive, then by Lemma 5.2 we obtain that the assumptions of Lemma 5.4 are satisfied, so using this lemma we obtain that all the numbers $\beta^{u-1, 0}, \dots, \beta^{u-1, v}$ in the Hom matrix, on the anti-diagonal just above the previous anti-diagonal, are positive. Put $u := u - 1$ and proceed with the paragraph under Step 3.

Note that this process terminates: For $-u$ large enough, the position corresponding to $\beta^{u, 0}$ will be a point on the boundary of the Hom matrix, so $\beta^{u, 0} = 0$.

Conclusion. In each case the parallelogram marked off by X' and X'' is of one of the types **(A)**, **(B)**, **(C)**, **(D)** or **(E)** in Section 4.1. (Namely, if none of the modules P_1^r is in the rectangle, then the type is **(A)**; if X'' is one of the P_1^r , then the type is **(B)**; in case $X' \cong P_1^r$ for some r , then the type is **(D)**; and if there are modules P_1^r (for $r \neq 1$) in the rectangle, but neither X' nor X'' has this form, then the type is **(C)**; finally if the module P_1^1 is in the rectangle, then the type is **(E)**.) Write $Z = Z_0 \oplus X' \oplus X''$ and let X_1, \dots, X_s ($s = 2$ or 3) be the modules corresponding to the entry -1 in the multiplicity matrix for the arc operation. Put $Z' = Z_0 \oplus X_1 \oplus \dots \oplus X_s$. By this construction, $Z' \leq_{\text{arc}} Z$. We obtain the arc diagram of Z' from the arc diagram of Z by a single move of type **(A)**, **(B)**, **(C)**, **(D)** or **(E)**, so the hom matrix was changed in the way described in Section 4.1, so we obtain that

$$\delta H(Y, Z')_X = \delta H(Y, Z)_X + \begin{cases} -1 & \text{if } X \text{ is in the parallelogram} \\ 0 & \text{otherwise} \end{cases}$$

Since the entries for $\delta H(Y, Z)$ within the parallelogram are all positive, the matrix $\delta H(Y, Z')$ is nonnegative and hence

$$Y \leq_{\text{hom}} Z' \quad \text{and} \quad Z' \leq_{\text{arc}} Z.$$

□

Proof of Theorem 6.1. Assume that $Y \leq_{\text{hom}} Z$. Since each application of Proposition 6.2 reduces the number of crossings in the arc diagram, a finite number of steps suffices to produce a sequence of modules $Z, Z', Z'', \dots, Z^{(m)} = Y$ such that

$$Y = Z^{(m)} \leq_{\text{arc}} Z^{(m-1)} \leq_{\text{arc}} \dots \leq_{\text{arc}} Z'' \leq_{\text{arc}} Z' \leq_{\text{arc}} Z.$$

□

6.2. An example

In this section we present an example illustrating the proof of Proposition 6.2. The arc diagrams



represent the modules $Y = B_2^{7,3} \oplus B_2^{6,2} \oplus P_2^5 \oplus P_0^4 \oplus P_1^1$ and $Z = B_2^{6,3} \oplus B_2^{5,1} \oplus P_1^7 \oplus P_1^4 \oplus P_1^2$.

We first compute the Hom-matrix to verify that $Y \leq_{\text{hom}} Z$. In this matrix, the entries on the first diagonal represent the numbers $\delta H_{B_2^{3,1}}, \delta H_{B_2^{4,1}}, \dots$. By Lemma 5.1, there are also zeros along the two top rows, which we indicate by solid lines.

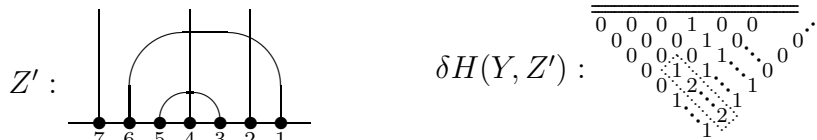
$$\delta H(Y, Z) : \begin{array}{cccccccc} \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \end{array}$$

We see that $Y \leq_{\text{hom}} Z$. We show that $Y \leq_{\text{arc}} Z$.

Following Steps 1,2,3 in the proof of Proposition 6.2, we choose the smaller parallelogram in the Hom-matrix, as indicated.

$$\delta H(Y, Z) : \begin{array}{cccccccc} \overline{0} & \overline{0} & \overline{0} & \overline{0} & \overline{1} & \overline{0} & \overline{0} & \overline{0} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 \end{array}$$

The corresponding operation of type **(A)** replaces the bipickets $B_2^{5,1}$ and $B_2^{6,3}$ in Z (representing arcs from 5 to 1 and from 6 to 3, respectively) by bipickets $B_2^{6,1}$ and $B_2^{5,3}$ in Z' , so Z' is given by the following arc diagram. We also indicate the new Hom-matrix.



Following Steps 1,2,3 in the proof of Proposition 6.2, we choose a parallelogram (alternatively, we can choose the parallelogram given by the upper

diagonal in the region marked by the 1's) and perform the operation indicated: Replace in Z' the bipicket $B_2^{6,1}$ and picket P_1^2 by bipicket $B_2^{6,2}$ and picket P_1^1 in Z'' . Arc diagram and Hom-matrix are as follows.

$$Z'' : \begin{array}{c} \text{Arc diagram with 7 points (1 to 7 from right to left). Arcs connect (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array} \quad \delta H(Y, Z'') : \begin{array}{c} \text{Hom-matrix with 7 rows and 7 columns. The main diagonal is 1's, and there are 1's at (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array}$$

The new parallelogram corresponds to move of a type **(E)**, so we replace the pickets P_1^7 and P_1^1 in Z'' by bipicket $B_2^{7,1}$ in Z''' . Arc diagram and Hom-matrix are as follows.

$$Z''' : \begin{array}{c} \text{Arc diagram with 7 points. Arcs connect (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array} \quad \delta H(Y, Z''') : \begin{array}{c} \text{Hom-matrix with 7 rows and 7 columns. The main diagonal is 1's, and there are 1's at (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array}$$

The parallelogram gives an operation of type **(B)**, namely we have to replace the bipicket $B_2^{5,3}$ and the picket P_1^4 in Z''' by pickets P_2^5 , P_0^4 and P_1^3 in Z^{IV} (note that the first two pickets represent an arc from 5 to 4, the last is a pole at 3).

$$Z^{IV} : \begin{array}{c} \text{Arc diagram with 7 points. Arcs connect (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array} \quad \delta H(Y, Z^{IV}) : \begin{array}{c} \text{Hom-matrix with 7 rows and 7 columns. The main diagonal is 1's, and there are 1's at (1,2), (2,3), (3,4), (4,5), (5,6), (6,7).} \end{array}$$

Finally, an operation of type **(B)** reduces the Hom-matrix to zero and yields the module Y : We replace the bipicket $B_2^{7,1}$ and the picket P_1^3 in Z^{IV} by the bipicket $B_2^{7,3}$ and the picket P_1^1 for Y . We are done:

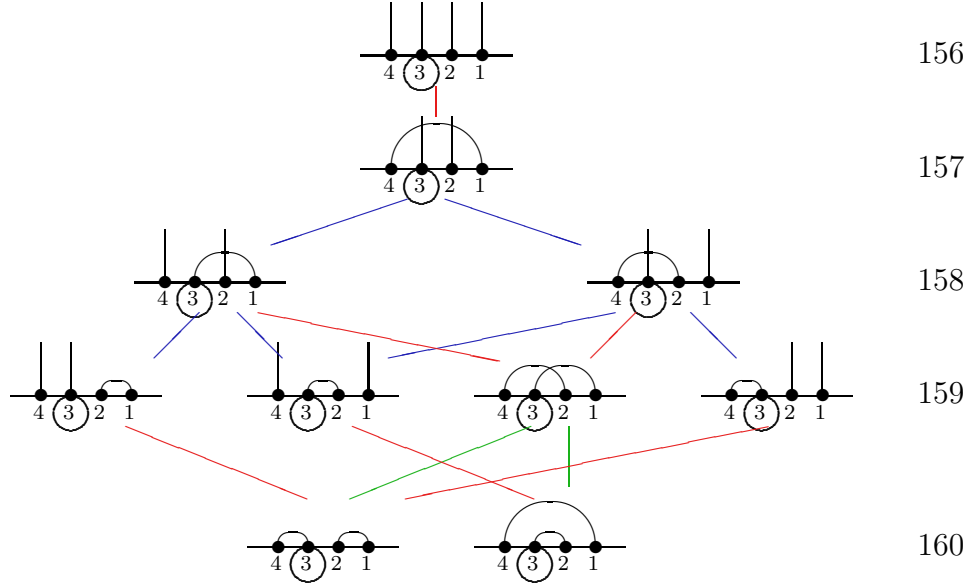
$$Y \leq_{\text{arc}} Z^{IV} \leq_{\text{arc}} Z''' \leq_{\text{arc}} Z'' \leq_{\text{arc}} Z' \leq_{\text{arc}} Z$$

7. The poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$

We discuss the maximal and minimal elements in the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$. First, we give an example.

7.1. An example

Example: We present the Hasse diagram of the poset $({}_2\mathcal{S}_{(3,2,1,1)}^{(4,3,3,2,1)}, \leq_{\text{arc}})$. The numbers given on the right hand side are the dimensions of \mathbb{V}_Δ for diagrams Δ in the same row. These dimensions are computed using Theorem 1.4.



7.2. Maximal and minimal elements

PROPOSITION 7.1. *Let γ, β be partitions.*

1. *In the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ there exists exactly one maximal element M . Its arc diagram has only poles and loops.*
2. *If $\beta \setminus \gamma$ is vertical strip, then in the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ there are exactly $c_{\alpha, \gamma}^\beta$ minimal elements, where $c_{\alpha, \gamma}^\beta$ is the Littlewood-Richardson coefficient, $\alpha = (2, 2, \dots, 2)$, if $|\beta| \setminus |\gamma|$ is even and $\alpha = (2, 2, \dots, 2, 1)$, if $|\beta| \setminus |\gamma|$ is odd. Its arc diagram has no crossings, has only arcs, at most one pole, and may contain loops.*

Proof. 1. We start with the observation that any object with an arc can be modified by the move **(E)**. In this way we obtain a bigger object in our poset. It follows that each maximal element in the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ has no arc. The location of loops in the arc diagram is determined by

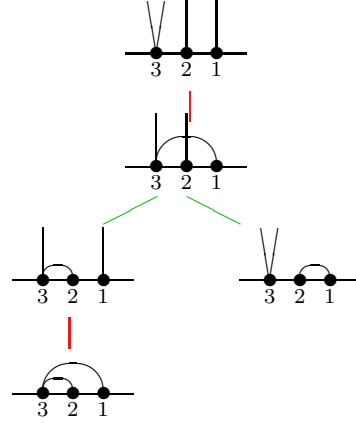
the partitions β and γ so there exists exactly one maximal object, it has the required properties.

2. If the tableau is a vertical strip, then in the arc diagram there is no double pole at any given point. Therefore, if there are at least two poles in an arc diagram, we can choose two poles in the points m and r such that $m \neq r$. It follows that, applying the move **(E)**, we can obtain a smaller object in our poset. We conclude that each minimal element in the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ has at most one pole. By [3, Theorem 5.7], in the poset $({}_2\mathcal{S}_\gamma^\beta, \leq_{\text{arc}})$ there are exactly $c_{\alpha,\gamma}^\beta$ minimal elements, where $c_{\alpha,\gamma}^\beta$ is the Littlewood-Richardson coefficient, $\alpha = (2, 2, \dots, 2)$, if $|\beta| \setminus |\gamma|$ is even and $\alpha = (2, 2, \dots, 2, 1)$, if $|\beta| \setminus |\gamma|$ is odd.

□

The following example shows that the assumption about the vertical strip in Statement 2 of Proposition 7.1 is necessary.

Example: The poset ${}_2\mathcal{S}_{(2,2,1)}^{(3,3,2,1)}$ looks as follows.



8. Two non-examples

In the paper we fixed two partitions, β and γ and considered the variety ${}_2\mathbb{V}_\gamma^\beta$. We present two examples which show that similar results cannot be obtained if we fix two other partitions, (that is, either α and γ or α and β).

8.1. Fixing partitions α and β

Example: Let α, β be partitions such that $\alpha_1 \leq 2$. We consider the full subcategory ${}_a\mathcal{S}^\beta$ of \mathcal{S}_2 consisting of all objects (N_α, N_β, f) . For $\alpha = (2)$ and $\beta = (m, r)$, where $m > r + 1$ we define two objects $X = B_2^{m,r}$ and $Y = P_0^m \oplus P_2^r$ in ${}_a\mathcal{S}^\beta$. Using the table from Section 6 we compute that

$$Y \leq_{\text{hom}} X.$$

The object X is indecomposable, so it is not true that:

$$Y \leq_{\text{ext}} X.$$

8.2. Fixing partitions α and γ

Example: Let α, γ be partitions such that $\alpha_1 \leq 2$. We consider the full subcategory ${}_a\mathcal{S}_\gamma$ of \mathcal{S}_2 consisting of all objects (N_α, N_β, f) , such that $\text{Coker } f \simeq N_\gamma$. For $\alpha = (2)$ and $\gamma = (m - 1, r - 1)$, where $m > r + 1$ we define two objects $X = B_2^{m,r}$ and $Y = P_2^{m+1} \oplus P_0^{r-1}$ in ${}_a\mathcal{S}_\gamma$. Using the table from section 6 we compute that

$$Y \leq_{\text{hom}} X.$$

The object X is indecomposable, so it is not true that:

$$Y \leq_{\text{ext}} X.$$

9. The dimensions of the orbits

In this section we present the proof of Theorem 1.4.

9.1. Previous results

We briefly recall notation and a result from [3].

We consider the affine variety $\mathbb{H}_\alpha^\beta(k) = \text{Hom}_k(N_\alpha(k), N_\beta(k))$ (consisting of all $|\beta| \times |\alpha|$ -matrices with coefficients in k). On $\mathbb{H}_\alpha^\beta(k)$ we consider the Zariski topology and on all subsets of $\mathbb{H}_\alpha^\beta(k)$ we work with the induced topology. Let $\mathbb{V}_{\alpha,\gamma}^\beta(k)$ be the subset of $\mathbb{H}_\alpha^\beta(k)$ consisting of all matrices that define a monomorphism $f : N_\alpha \rightarrow N_\beta$ in the category $\mathcal{N}(k)$ with $\text{Coker } f \simeq N_\gamma$. On $\mathbb{V}_{\alpha,\gamma}^\beta(k)$ acts the algebraic group $\text{Aut}_{\mathcal{N}}(N_\alpha(k)) \times \text{Aut}_{\mathcal{N}}(N_\beta(k))$ via $(g, h) \cdot f =$

hfg^{-1} . The orbits of this action correspond bijectively to isomorphism classes of objects in $\mathcal{S}_{\alpha,\gamma}^\beta$. For a map $f : N_\alpha(k) \rightarrow N_\beta(k)$, denote by G_f the orbit of f in $\mathbb{V}_{\alpha,\gamma}^\beta(k)$.

We recall a theorem which yields information about the dimensions of orbits in $\mathbb{V}_{\alpha,\gamma}^\beta(k)$. This theorem was proved in [3, Section 5].

THEOREM 9.1. *Let k be an algebraically closed field, and let α, β, γ be partitions. Suppose the arc diagram Δ of an invariant subspace $Y = (N_\alpha, N_\beta, f) \in \mathbb{V}_{\alpha,\gamma}^\beta(k)$ has $x(\Delta)$ crossings. Then*

$$\dim G_f = \deg h_{\alpha,\gamma}^\beta + \deg a_\alpha - x(\Delta),$$

where $\deg h_{\alpha,\gamma}^\beta = n(\beta) - n(\alpha) - n(\gamma)$ is the degree of the Hall polynomial $h_{\alpha,\gamma}^\beta(q)$ and $\deg a_\alpha = |\alpha| + 2n(\alpha)$ is the degree of the polynomial $a_\alpha(q)$ which counts the automorphisms of $N_\alpha(\mathbb{F}_q)$. \square

9.2. The dimensions of the orbits in ${}_2\mathbb{V}_\gamma^\beta$.

For a group G acting on a variety X and for $y \in X$, we denote by $G.y$ the stabilizer of y in G .

Proof of Theorem 1.4. Let $Y = (N_\alpha, N_\beta, f) \in {}_2\mathcal{S}_\gamma^\beta$ and let $F = (\varphi_\alpha, f, \varphi_\beta) \in {}_2\mathbb{V}_\gamma^\beta$ corresponds to Y . Denote by Δ the arc diagram diagram corresponding to Y and assume that Δ has $x(\Delta)$ crossings.

It is well known that

$$(9.2) \quad \dim \mathcal{O}_F = \dim Gl(a, b) - \dim Gl(a, b).F$$

where $a = |\alpha|$ and $b = |\beta|$. Moreover

$$(9.3) \quad \dim \text{Aut}(Y) = \dim Gl(a, b).F.$$

Similarly,

$$\begin{aligned} \dim G_f &= \dim \text{Aut}_{\mathcal{N}}(N_\alpha(k)) \times \text{Aut}_{\mathcal{N}}(N_\beta(k)) \\ &\quad - \dim(\text{Aut}_{\mathcal{N}}(N_\alpha(k)) \times \text{Aut}_{\mathcal{N}}(N_\beta(k))).f. \end{aligned}$$

and

$$(9.4) \quad \dim \text{Aut}(Y) = \dim(\text{Aut}_{\mathcal{N}}(N_\alpha(k)) \times \text{Aut}_{\mathcal{N}}(N_\beta(k))).f.$$

Substituting 9.3 into 9.2 we obtain

$$(9.5) \quad \dim \mathcal{O}_F = \dim Gl(a, b) - \dim \text{Aut}(Y)$$

and similarly for 9.4 we have

$$(9.6) \quad \dim G_f = \dim \text{Aut}_{\mathcal{N}}(N_{\alpha}(k)) \times \text{Aut}_{\mathcal{N}}(N_{\beta}(k)) - \dim \text{Aut}(Y).$$

Combining 9.5 with 9.6 yields

$$(9.7) \quad \dim \mathcal{O}_F = \dim Gl(a, b) - (\dim \text{Aut}_{\mathcal{N}}(N_{\alpha}(k)) \times \text{Aut}_{\mathcal{N}}(N_{\beta}(k)) - \dim G_f).$$

Finally, we obtain from Theorem 9.1

$$\dim \mathcal{O}_F = |\beta|^2 + |\alpha|^2 - n(\alpha) - n(\beta) - n(\gamma) - |\beta| - x(\Delta).$$

□

9.3. The effect of a single arc move

We finish by presenting an example which shows that there is no upper bound on the change of the orbit dimension resulting from a single arc move.

Example: Let $\beta = (3, 1, 1, \dots, 1), \gamma = (2)$. Then in the category $\mathcal{S}_{\gamma}^{\beta}$ we have only two objects $X = B_2^{3,1} \oplus \bigoplus_{i=1}^n P_1^1$ and $Y = P_1^3 \oplus \bigoplus_{i=1}^{n+1} P_1^1$, up to isomorphism. They have the following arc diagrams:



Note that we can obtain the diagram $\Delta(Y)$ from the diagram $\Delta(X)$ by a single move **(E)**. Using Theorem 1.4, we compute

$$\begin{aligned} \dim \mathbb{V}_{\Delta(X)} &= |\beta|^2 + |\alpha|^2 - (2 \cdot 0 + 1 \cdot 1 + 1 \cdot 2 + \dots + 1 \cdot n) \\ &\quad - n(\beta) - n(\gamma) - |\beta| - 0 \\ &= |\beta|^2 + |\alpha|^2 - (1 \cdot 0 + 1 \cdot 1 + 1 \cdot 2 + \dots + 1 \cdot n + 1 \cdot (n+1)) \\ &\quad - n(\beta) - n(\gamma) - |\beta| - 0 \quad + (n+1) \\ &= \dim \mathbb{V}_{\Delta(Y)} \quad + (n+1). \end{aligned}$$

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